Example 4.1 Calculation of Open Rational B-spline Curves

Consider the control polygon given by the vertices

$$B_1\,[0 \quad 0]\,,\; B_2\,[1 \quad 2]\,,\; B_3\,[5/2 \quad 0]\,,\; B_4\,[4 \quad 2]\,,\; B_5\,[5 \quad 0]$$

Determine the point at $t = 3/2$ for the third-order ($k = 3$) open rational B-spline curve with homogeneous weighting factors given by $[\,H\,] = [\,1 \quad 1 \quad h_3 \quad 1 \quad 1\,]$, $h_3 = 0, 1/4, 1, 5$.

The knot vector is $[\,0 \quad 0 \quad 0 \quad 1 \quad 2 \quad 3 \quad 3 \quad 3\,]$. The parameter range is $0 \le t \le 3$. The curves are composed of three piecewise rational quadratics, one for each of the interior intervals in the knot vector.

Using Eqs. (3.2) on the interval $1 \le t < 2$, the nonrational B-spline basis functions are

$$1 \le t < 2$$

$$N_{4,1}(t) = 1; \qquad N_{i,1}(t) = 0, \quad i \neq 4$$

$$N_{3,2}(t) = (2-t); \qquad N_{4,2}(t) = (t-1); \qquad N_{i,2}(t) = 0, \quad i \neq 3, 4$$

$$N_{2,3}(t) = \frac{(2-t)^2}{2}; \qquad N_{3,3}(t) = \frac{t(2-t)}{2} + \frac{(3-t)(t-1)}{2}$$

$$N_{4,3}(t) = \frac{(t-1)^2}{2}; \qquad N_{i,3}(t) = 0, \quad i \neq 2, 3, 4$$

From Eq. (4.3) and these results, after first determining the denominator

$$S = \sum_{i=1}^{n+1} h_i N_{i,k}(t) = h_1 N_{1,3}(t) + h_2 N_{2,3}(t) + h_3 N_{3,3}(t) + h_4 N_{4,3}(t) + h_5 N_{5,3}(t)$$

the rational B-spline basis functions are

$$1 \le t < 2$$

$$h_3 = 0$$

$$S = h_2 N_{2,3}(t) + h_4 N_{4,3}(t)$$

$$= \frac{(2-t)^2}{2} + \frac{(t-1)^2}{2} = \frac{2t^2 - 6t + 5}{2}$$

$$R_{1,3}(t) = 0$$

$$R_{2,3}(t) = \frac{h_2 N_{2,3}(t)}{S} = \frac{(2-t)^2}{2t^2 - 6t + 5}$$

$$R_{3,3}(t) = 0$$

$$R_{4,3}(t) = \frac{h_4 N_{4,3}(t)}{S} = \frac{(t-1)^2}{2t^2 - 6t + 5}$$

$$R_{5,3}(t) = 0$$

$h_3 = 1/4$

$$S = h_2 N_{2,3}(t) + h_3 N_{3,3}(t) + h_4 N_{4,3}(t)$$
$$= \frac{(2-t)^2}{2} + \frac{t(2-t)}{8} + \frac{(3-t)(t-1)}{8} + \frac{(t-1)^2}{2}$$
$$= \frac{6t^2 - 18t + 17}{8}$$

$$R_{1,3}(t) = 0$$
$$R_{2,3}(t) = \frac{4(2-t)^2}{6t^2 - 18t + 17}$$
$$R_{3,3}(t) = \frac{t(2-t) + (3-t)(t-1)}{6t^2 - 18t + 17} = \frac{-2t^2 + 6t - 3}{6t^2 - 18t + 17}$$
$$R_{4,3}(t) = \frac{4(t-1)^2}{6t^2 - 18t + 17}$$
$$R_{5,3}(t) = 0$$

$h_3 = 1$

$$S = 1$$
$$R_{1,3}(t) = 0$$
$$R_{2,3}(t) = N_{2,3}(t) = \frac{(2-t)^2}{2}$$
$$R_{3,3}(t) = N_{3,3}(t) = \frac{t(2-t)}{2} + \frac{(3-t)(t-1)}{2}$$
$$R_{4,3}(t) = N_{4,3}(t) = \frac{(t-1)^2}{2}$$
$$R_{5,3}(t) = 0$$

$h_3 = 5$

$$S = \frac{(2-t)^2}{2} + \frac{5t(2-t)}{2} + \frac{5(3-t)(t-1)}{2} + \frac{(t-1)^2}{2}$$
$$= -4t^2 + 12t - 5$$

$$R_{1,3}(t) = 0$$
$$R_{2,3}(t) = \frac{(2-t)^2}{2(-4t^2 + 12t - 5)}$$
$$R_{3,3}(t) = \frac{5t(2-t) + 5(3-t)(t-1)}{2(-4t^2 + 12t - 5)} = \frac{5(-2t^2 + 6t - 3)}{2(-4t^2 + 12t - 5)}$$
$$R_{4,3}(t) = \frac{(t-1)^2}{2(-4t^2 + 12t - 5)}$$
$$R_{5,3}(t) = 0$$

Complete results are shown in Fig. 4.1.

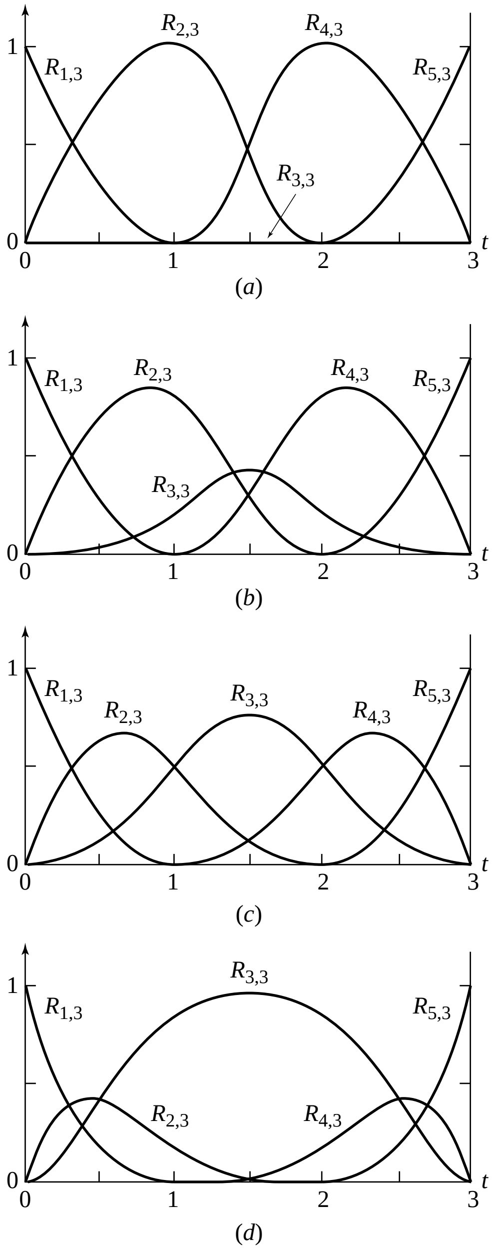

Figure 4.1 Rational B-spline basis functions for $n+1=5$, $k=3$ with open knot vector $[X]=[0\ \ 0\ \ 0\ \ 1\ \ 2\ \ 3\ \ 3\ \ 3]$, $[H]=[1\ \ 1\ \ h_3\ \ 1\ \ 1]$. (a) $h_3=0$; (b) $h_3={}^1\!/_4$; (c) $h_3=1$; (d) $h_3=5$.

Evaluating these results at $t = 3/2$ yields

$$
\begin{aligned}
h_3 = 0: \quad & R_{1,3}(3/2) = 0; \quad R_{2,3}(3/2) = 1/2; \quad R_{3,3}(3/2) = 0; \\
& R_{4,3}(3/2) = 1/2; \quad R_{5,3}(3/2) = 0 \\
h_3 = 1/4: \quad & R_{1,3}(3/2) = 0; \quad R_{2,3}(3/2) = 2/7; \quad R_{3,3}(3/2) = 3/7; \\
& R_{4,3}(3/2) = 2/7; \quad R_{5,3}(3/2) = 0 \\
h_3 = 1: \quad & R_{1,3}(3/2) = 0; \quad R_{2,3}(3/2) = 1/8; \quad R_{3,3}(3/2) = 3/4; \\
& R_{4,3}(3/2) = 1/8; \quad R_{5,3}(3/2) = 0 \\
h_3 = 5: \quad & R_{1,3}(3/2) = 0; \quad R_{2,3}(3/2) = 1/32; \quad R_{3,3}(3/2) = 15/16; \\
& R_{4,3}(3/2) = 1/32; \quad R_{5,3}(3/2) = 0
\end{aligned}
$$

The corresponding points on the rational B-spline curves are

$$
\begin{aligned}
h_3 = 0: \quad & P(3/2) = 1/2\,[1 \quad 2] + 1/2\,[4 \quad 2] = [5/2 \quad 2] \\
h_3 = 1/4: \quad & P(3/2) = 2/7\,[1 \quad 2] + 3/7\,[5/2 \quad 0] + 2/7\,[4 \quad 2] = [5/2 \quad 8/7] \\
h_3 = 1: \quad & P(3/2) = 1/8\,[1 \quad 2] + 3/4\,[5/2 \quad 0] + 1/8\,[4 \quad 2] = [5/2 \quad 1/2] \\
h_3 = 5: \quad & P(3/2) = 1/32\,[1 \quad 2] + 15/16\,[5/2 \quad 0] + 1/32\,[4 \quad 2] = [5/2 \quad 1/8]
\end{aligned}
$$

Complete results are shown in Fig. 4.2.

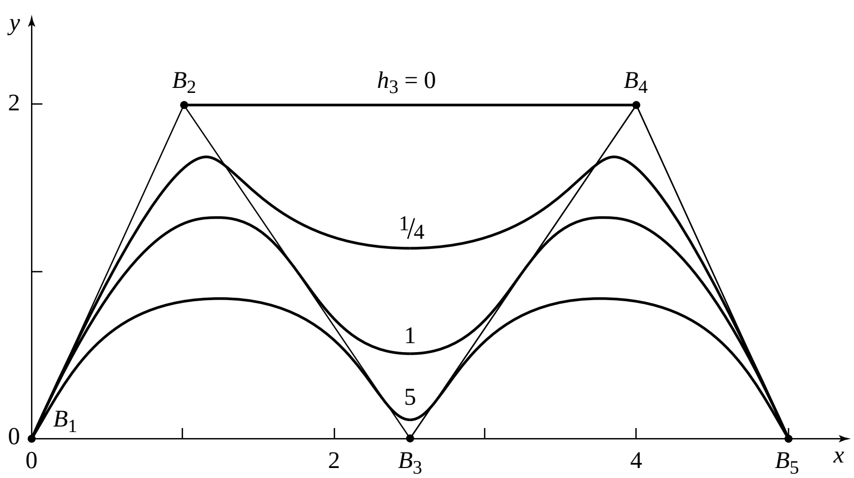

Figure 4.2 Rational B-spline curves for $n+1=5$, $k=3$ with open knot vector $[X] = [0 \quad 0 \quad 0 \quad 1 \quad 2 \quad 3 \quad 3 \quad 3]$ and $[H] = [1 \quad 1 \quad h_3 \quad 1 \quad 1]$.