TWO

EXACT SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

2–1 Introduction

Because of the great complexity of the full compressible Navier-Stokes equations, no known general analytical solution exists. Hence, it is necessary to simplify the equations either by making assumptions about the fluid, about the flow or about the geometry of the problem in order to obtain analytical solutions. Typical assumptions are that the flow is laminar, steady, two-dimensional, the fluid incompressible with constant properties and the flow is between parallel plates. By so doing it is possible to obtain analytical, exact and approximate solutions to the Navier-Stokes equations.

Before proceeding let us clearly define what is meant by analytical, exact and approximate solutions. An analytical solution is obtained when the governing boundary value problem is integrated using the methods of classical differential equations. The result is an algebraic expression giving the dependent variable(s) as a function(s) of the independent variable(s). An exact solution is obtained by integrating the governing boundary value problem numerically. The result is a tabulation of the dependent variable(s) as a function(s) of the independent variables(s). An approximate solution results when methods such as series expansion and the von Karman-Pohlhausen technique are used to solve the governing boundary value problem (see Schlichting [Schl60], p. 239).

2–2 Analytical Solutions

Finding analytical solutions of the Navier-Stokes equations, even in the uncoupled case (see Section 1–10), presents almost insurmountable mathematical difficulties due to the nonlinear character of the equations. However, it is possible to find analytical solutions in certain particular cases, generally when the nonlinear convective terms vanish naturally. Parallel flows, in which only one velocity component is different from zero, of a two-dimensional, incompressible fluid have this characteristic. Examples for which analytical solutions exist are parallel flow through a straight channel, Couette flow and Hagen-Poiseuille flow, i.e., flow in a cylindrical pipe. Here we discuss parallel flow through a straight channel and Couette flow.

2–3 Parallel Flow Through A Straight Channel

A flow is considered parallel if only one component of the velocity is different from zero. In order to illustrate this concept, consider two-dimensional steady flow in a channel with straight parallel sides (see Figure 2–1). This flow is two-dimensional since the velocity is in the x direction and, as we shall see, its variation is in the y direction. We consider the fluid to be incompressible and to have constant properties. Under these circumstances the momentum and energy equations are uncoupled (see Section 1–10). Thus, we consider only the continuity and the momentum equations, i.e., the velocity field, and reserve our discussion of the energy equation, i.e., the temperature field, until Chapter 4. The continuity and momentum equations for two-dimensional, steady, incompressible, constant property flow are

continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1-49)$$

x momentum

$$u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = -\frac{1}{\rho}\,\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad (1-50a)$$

y momentum

$$u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} = -\frac{1}{\rho}\,\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \qquad (1-50b)$$

Since the flow is constrained by the flat parallel walls of the channel, no component of the velocity in the y direction is possible, i.e., $v = 0$. This implies

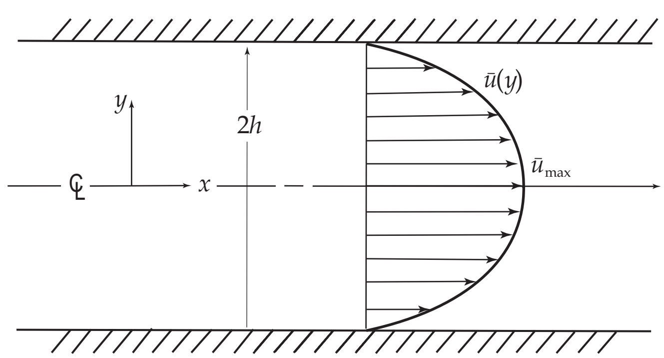

Figure 2–1. Parallel flow through a straight channel.

that $v = 0$ everywhere. Hence, the gradients in v are also equal to zero, i.e., $\partial v/\partial y = \partial v/\partial x = \partial^2 v/\partial y^2 = \partial^2 v/\partial x^2 = 0$.

From the continuity equation we have

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} = 0$$

which implies that $\partial u^2/\partial x^2 = 0$. The momentum equations are thus reduced to

$$\frac{dP}{dx} = \mu \frac{d^2 u}{dy^2} \qquad (2-1)$$

and

$$\frac{\partial P}{\partial y} = 0$$

Since u is not a function of x, the distance along the axis, there is no physical mechanism to provide a change in the pressure gradient. Thus, the pressure gradient is considered constant, i.e., dP/dx = constant. Note that under these conditions all of the nonlinear convective terms in the momentum equations are eliminated. Equation (2–1) is a linear second-order ordinary differential equation. It is easily integrated twice to yield

$$u(y) = \frac{1}{2}\frac{1}{\mu}\frac{dP}{dx}y^2 + Ay + B \qquad (2-2)$$

where A and B are integration constants.

In order to evaluate the integration constants, we apply the no-slip boundary conditions at the channel walls. The boundary conditions at the walls are

$$y = \pm h \qquad u = 0 \qquad (2-3)$$

Using these boundary conditions to evaluate the integration constants, we have

$$A = 0$$

$$B = -\frac{1}{2}\frac{h^2}{\mu}\frac{dP}{dx}$$

Substitution into Eq. (2–2) yields

$$u(y) = -\frac{1}{2}\frac{h^2}{\mu}\frac{dP}{dx}\left[1 - \left(\frac{y}{h}\right)^2\right] \qquad (2-4)$$

Here we see that the velocity distribution in the channel is parabolic and symmetrical about the axis. The maximum velocity, which occurs at the center of the channel, is given by

$$u_m = -\frac{1}{2}\frac{h^2}{\mu}\frac{dP}{dx} \qquad (2-5)$$

Introducing nondimensional variables

$$\bar{u} = \frac{u}{u_m} \qquad \bar{y} = \frac{y}{h}$$

yields

$$\bar{u} = 1 - \bar{y}^2 \tag{2-6}$$

This nondimensional velocity distribution is shown in Figure 2–1. Using Newton's law of friction, which is obtained with the help of Eqs. (1–15a) and (1–25), the shearing stress at the channel walls is given by

$$\tau_{(y=\pm h)} = \mu \frac{du}{dy}\bigg|_{(y=\pm h)} \tag{2-7}$$

or using nondimensional variables

$$\tau_{(\bar{y}=\pm 1)} = \mu \frac{u_m}{h} \frac{d\bar{u}}{d\bar{y}}\bigg|_{(\bar{y}=\pm 1)} \tag{2-8}$$

Thus, using Eqs. (2–5) and (2–6), the shearing stress at the channel walls is

$$\tau_{(\bar{y}=\pm 1)} = \pm h \frac{dP}{dx} \tag{2-9}$$

2–4 Couette Flow

Continuing our discussion of the analytical solutions, consider the flow between two parallel infinite flat surfaces, one of which is moving in its plane with a velocity U (see Figure 2–2). The flow is considered steady, two-dimensional and incompressible with constant properties. Using the same physical and geometric arguments presented in the previous discussion of channel flow, the governing equations again reduce to

$$\frac{dP}{dx} = \mu \frac{d^2 u}{dy^2} \tag{2-1}$$

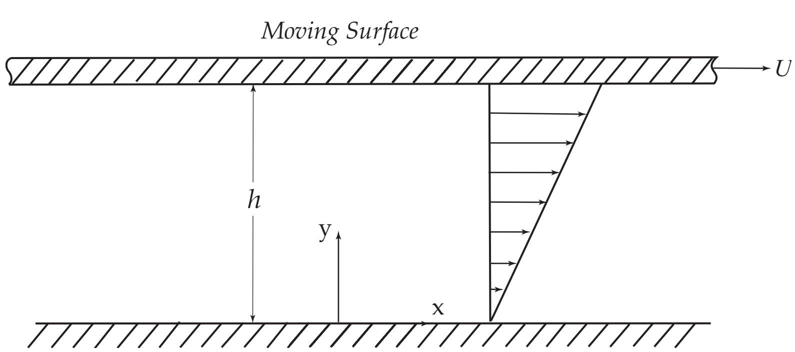

Figure 2–2. Couette flow.

No-slip boundary conditions are assumed to apply at both the moving and the stationary surfaces, i.e.,

$$y = 0 \quad u = 0 \qquad y = h \quad u = U \tag{2-10a, b}$$

For simple Couette flow the pressure gradient is assumed zero. Thus, the governing equation becomes

$$\frac{d^2u}{dy^2} = 0$$

Integrating twice and evaluating the constants of integration from the boundary conditions yields the linear velocity distribution shown in Figure 2–2.

$$\frac{u}{U} = \frac{y}{h} \quad \text{or} \quad \bar{u} = \bar{y} \tag{2-11}$$

Turning now to nonsimple Couette flow, i.e., when the pressure gradient is nonzero, the governing differential equation is now Eq. (2–1) with the boundary conditions given in Eqs. (2–10a, b). Integrating Eq. (2–1) twice again yields

$$u(y) = \frac{1}{2}\frac{1}{\mu}\frac{dP}{dx}y^2 + Ay + B \tag{2-2}$$

where A and B are constants of integration. Again, the pressure gradient is constant. Using the boundary conditions to evaluate A and B yields

$$B = 0$$
$$A = \frac{1}{h}\left(U - \frac{1}{2}\frac{1}{\mu}\frac{dP}{dx}h^2\right)$$

Substituting into Eq. (2–2), introducing the nondimensional variables

$$\bar{u} = \frac{u}{U} \qquad \bar{y} = \frac{y}{h} \qquad \bar{P} = -\frac{h^2}{2\mu U}\frac{dP}{dx} \tag{2-12a, b, c}$$

and rearranging yields

$$\bar{u} = \bar{y}\,[\,1 + \bar{P}(1 - \bar{y})\,] \tag{2-13}$$

From Eq. (2–13) we see that the shape of the nondimensional velocity distribution is determined by the nondimensional pressure gradient, $\bar{P}$. Nondimensional velocity profiles for several values of $\bar{P}$ are shown in Figure 2–3.

The results shown in Figure 2–3 indicate that the slope of the velocity profile

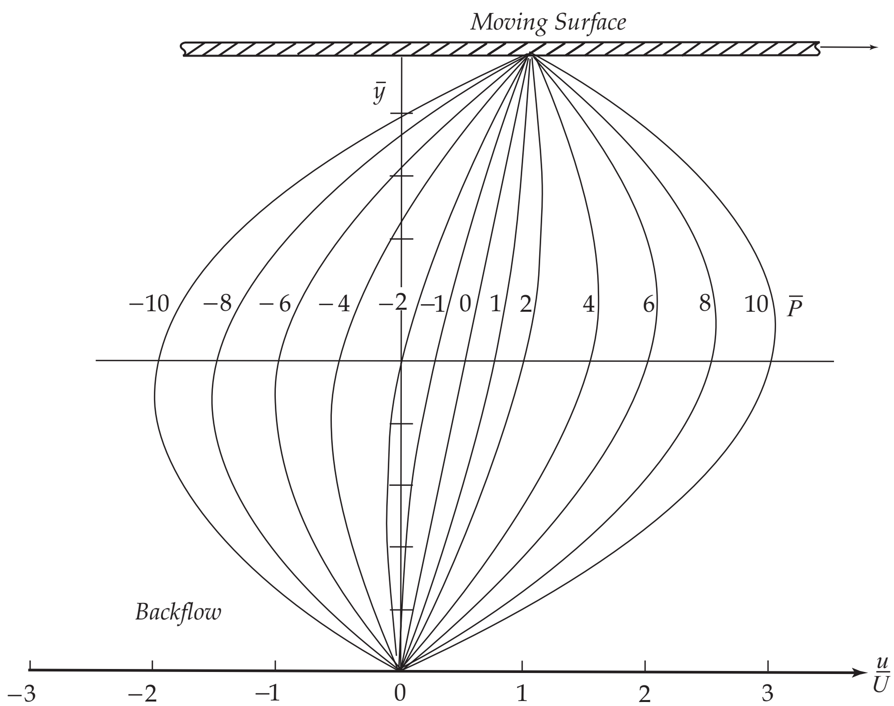

Figure 2–3. Nondimensional velocity profiles for nonsimple Couette flow.

Hence, the shearing stress at the stationary surface is zero for $\bar{P} = -1$ and at the moving surface for $\bar{P} = +1$. Using Eqs. (2–13) and (2–14) we see that for $\bar{P} < -1$, regions of backflow, i.e., $u \le 0$, exist near the stationary surface. For $\bar{P} > 1$, the velocity in the flow exceeds the velocity of the moving plate. Physically, a region of backflow exists when the force due to the momentum of the fluid in the flow direction is overcome by the adverse pressure gradient† in the flow direction. Similarly, velocities greater than that of the moving plate occur when a favorable pressure gradient in the flow direction adds to the momentum of the fluid in that direction.

One further analytical solution, the suddenly accelerated plane wall, is presented below in order to illustrate several solution techniques.

2–5 The Suddenly Accelerated Plane Wall

Consider the two-dimensional parallel flow of an incompressible fluid near a flat plate which is suddenly accelerated from rest and moves in its own plane with

†An adverse pressure gradient means the pressure *increases* in the flow direction.

a constant velocity U (see Figure 2–4). Recall that the Navier-Stokes equations for two-dimensional incompressible, constant property flow are

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = -\frac{1}{\rho}\,\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} \overset{\nu}{+} \left(\partial^2 u \right.\right.$$

which requires $f(x) = \text{constant}$. The mathematical model is then based upon a uniform pressure field for all x, y and t equal to or greater than zero.

From the continuity equation we have

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} = 0$$

and the governing differential equations reduce to

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \qquad t > 0 \tag{2-16}$$

The initial and boundary conditions are

$$\begin{array}{lll} u = 0 & \text{for all } y & t \leq 0 \\ \left.\begin{array}{ll} u = U & y = 0 \\ u \to 0 & y \to \infty \end{array}\right\} & & t > 0 \end{array} \tag{2-17a, b}$$

where for $t > 0$ the first boundary condition expresses the fact that there is no slip at the surface. The second boundary condition is read: In the limit as y approaches infinity the velocity approaches zero. It represents the physical condition that the influence of the wall, i.e., the effect of viscosity, decreases asymptotically to zero as the distance above the plate increases.

The governing differential equation, Eq. (2–16), has the same form as the equation describing the diffusion of heat by conduction in the space $y > 0$ when at $t = 0$ the wall temperature is suddenly changed. The kinematic viscosity, ν, which appears in the governing equation is sometimes called the momentum diffusivity. It plays the same role in the momentum equation that the thermal diffusivity, α, does in the energy equation. Equation (2–16) is generally called the diffusion equation. The diffusion equation is first encountered in a classical course in differential equations. There, an analytical solution is obtained by the classical separation of variables technique. Thus, it seems reasonable to look for a solution in the same manner. In an attempt to separate the variables we assume

$$u = F(t)G(y) \tag{2-18}$$

Substituting into the differential equation yields

$$F'G = \nu F G''$$

or

$$\frac{F'}{F} = \nu \frac{G''}{G} = i\gamma^2 \tag{2-19}$$

where here the prime denotes differentiation with respect to the appropriate argument. Since each side of Eq. (2–19) is a function of only one variable, they can individually be satisfied only if they are each equal to a constant, in this case $i\gamma^2$.[†] Hence, we have

$$\frac{F'}{F} = i\gamma^2 \tag{2-20a}$$

$$\frac{G''}{G} = \frac{i\gamma^2}{\nu} \tag{2-20b}$$

The solution of Eq. (2–20a) is

$$F = C_1 e^{i\gamma^2 t} \tag{2-21a}$$

and to Eq. (2–20b)

$$G = C_2 e^{\left(\frac{i\gamma^2}{\nu}\right)^{\frac{1}{2}} y} + C_3 e^{-\left(\frac{i\gamma^2}{\nu}\right)^{\frac{1}{2}} y} \tag{2-21b}$$

The initial and boundary conditions are given by Eq. (2–17). There are three possible cases, $i = 0, \pm 1$. The case of $i = 0$ yields the trivial solution $u = 0$ everywhere and hence is discarded. For $i = \pm$

We now seek a similarity solution. In particular, we seek a transformation of variables which reduces the governing partial differential equation to an ordinary differential equation. Since a partial differential equation involves more than one independent variable and an ordinary differential equation only one, it is reasonable to assume an independent variable transformation which attempts to combine the two independent variables. Thus, we assume

$$\eta = By^m t^n \qquad (2-22)$$

where B, m, n are, as yet, undetermined constants and η is a transformed independent variable. In addition, to nondimensionalize the equations we assume a dependent variable transformation of the form

$$u = Af(\eta) \qquad (2-23)$$

where A is again an as yet undetermined constant. Using these transformations yields

$$\frac{\partial u}{\partial t} = A\frac{\partial \eta}{\partial t}\frac{\partial f}{\partial \eta} = ABny^m t^{n-1} f' \qquad (2-24)$$

where here the prime denotes differentiation with respect to η. Further

$$\frac{\partial u}{\partial y} = A\frac{\partial \eta}{\partial y} f' = ABmy^{m-1} t^n f' \qquad (2-25a)$$

and

$$\frac{\partial^2 u}{\partial y^2} = ABm(m-1)y^{m-2}t^n f' + AB^2m^2y^{2(m-1)}t^{2n}f'' \qquad (2-25b)$$

Substituting Eqs. (2–24) and (2–25) into the differential equation, (Eq. 2-16), yields

$$ABny^m t^{n-1} f' = \nu ABm(m-1)y^{m-2}t^n f' + \nu AB^2m^2y^{2(m-1)}t^{2n}f'' \qquad (2-26)$$

Since A appears in each term it can be eliminated. Hence, its value is arbitrary *with respect to the differential equation.*

We now seek to select values of m, n, A, B such that reduction to a nondimensional ordinary differential equation is achieved. Inspection of Eq. (2–26) shows that for $m = 1$ the first term on the right and the y dependence of the second term are eliminated. Thus, we have

$$Bnyt^{(n-1)} f' = \nu B^2 t^{2n} f'' \qquad (2-27)$$

and

$$\eta = Byt^n \qquad (2-28)$$

Using Eq. (2–28) allows Eq. (2–27) to be rewritten as

$$n\eta t^{-1} f' = \nu B^2 t^{2n} f'' \qquad (2-29)$$

If $n = -1/2$ the time dependence is eliminated. Finally, choosing $B^2 = 1/4\,\nu$, where the 4 is introduced for later convenience, yields a nondimensional ordinary differential equation

$$f'' + 2\eta f' = 0 \tag{2-30}$$

where

$$\eta = \frac{1}{2}\frac{y}{\sqrt{\nu t}} \tag{2-31}$$

$$u = Af(\eta) \tag{2-32}$$

Now looking at the boundary conditions we have for $t > 0$, $\eta = 0$ when $y = 0$ and $\eta \to \infty$ when $y \to \infty$. Hence

$$\eta = 0 \qquad f(0) = \frac{U}{A} \tag{2-33}$$

$$\eta \to \infty \qquad f(\eta) \to 0 \tag{2-34}$$

The first of these boundary conditions, Eq. (2–33), is inconvenient in its present form. Thus, we take $A = U$ and have

$$\eta = 0 \qquad f(0) = 1 \tag{2-35}$$

Note that the arbitrariness of A as revealed by the differential equation is used to achieve a simplified boundary condition.

A closed-form analytical solution to the boundary value problem given by Eqs. (2–30), (2–34) and (2–35) is obtained by letting

$$\phi = \frac{df}{d\eta} = f' \tag{2-36}$$

Upon substitution into Eq. (2–30) we have

$$\phi' + 2\eta\phi = 0 \tag{2-37}$$

The solution of this ordinary differential equation is (see [Murp60], Eq. 173, p. 327)

$$\phi = f' = C_1 e^{-\eta^2} \tag{2-38}$$

Hence, after integrating

$$f = C_1 \int_0^{\eta} e^{-\eta^2}\, d\eta + C_2 \tag{2-39}$$

The boundary condition at $\eta = 0$ yields $C_2 = 1$. The boundary condition as $\eta \to \infty$ yields

$$C_1 = \frac{-1}{\displaystyle\int_0^{\infty} e^{-\eta^2}\, d\eta} \tag{2-40}$$

The solution is then

$$f(\eta) = 1 - \frac{\displaystyle\int_0^\eta e^{-\eta^2}\,d\eta}{\displaystyle\int_0^\infty e^{-\eta^2}\,d\eta} \qquad (2-41)$$

Evaluation of the integral from zero to infinity yields

$$f(\eta) = 1 - \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-\eta^2}\,d\eta \qquad (2-42)$$

The second term on the right is the error function and $1 - \mathrm{erf}(\eta)$ is the complementary error function. Thus,

$$f(\eta) = \frac{u}{U} = \mathrm{erfc}(\eta) \qquad (2-43)$$

Here, a similarity analysis has been used to obtain a closed-form analytical solution to a problem which previously did not yield to any solution using separation of variables.

Figure 2–5a gives y vs u/U for various values of time t. Note that the extent of the viscous zone increases with increasing time. Due to the action of fluid viscosity, after an infinite time the entire flow field above the plate is moving with the velocity of the plate. Figure 2–5b shows η vs u/U. This figure illustrates that the similarity variables collapse the solutions given in Figure 2–5a into a single solution.

An exact solution to this boundary value problem can be obtained by numerical integration and compared with the analytical solution to illustrate the

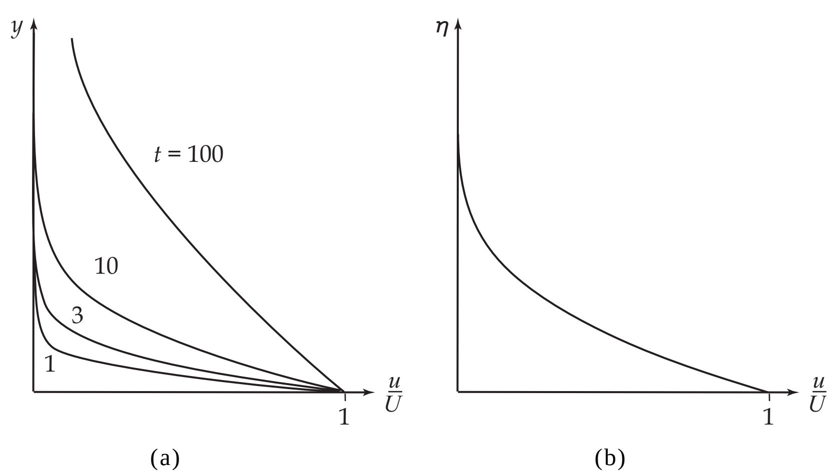

Figure 2–5. Velocity profiles for the suddenly accelerated plate.

accuracy of numerical integration techniques. Having demonstrated the utility of similarity analysis, we proceed to apply this technique to problems which generally do not yield closed-form analytical solutions (see Hansen [Hans64]).

2–7 Two-dimensional Stagnation Point Flow

Consider the two-dimensional steady flow of an incompressible viscous nonheat-conducting fluid impinging on a plate perpendicular to the flow direction (see Figure 2–6). Assume that the flow at a large distance above the plate is given by the corresponding inviscid (potential) flow and that no-slip conditions prevail at the plate surface.

First, consider the inviscid flow solution. We take the plate to be at $y = 0$ and the stagnation point at $x = 0$, $y = 0$ (see Figure 2–6). The flow is impinging on the plate from the positive y direction. Under these circumstances the potential flow solution yields the following expressions for the stream and velocity potential functions (see [Rose63], p. 155)

$$\psi_i = K\,xy \tag{2-44}$$

$$\phi_i = \frac{K}{2}\,(x^2 - y^2) \tag{2-45}$$

where the i subscript indicates the inviscid flow solution. Differentiation yields U_i and V_i, the x and y components of the inviscid flow velocity, i.e.,

$$U_i = \frac{\partial \psi_i}{\partial y} = K\,x \tag{2-46}$$

$$V_i = -\frac{\partial \psi_i}{\partial x} = -K\,y \tag{2-47}$$

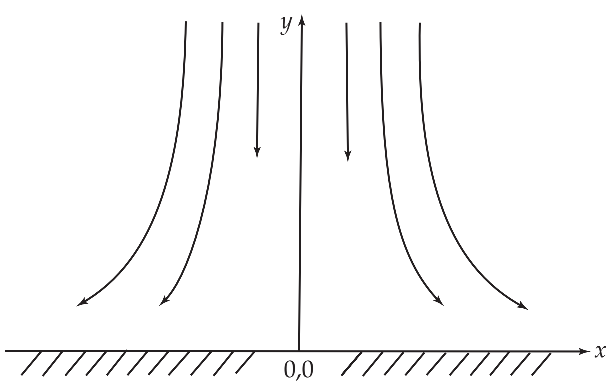

Figure 2–6. Two-dimensional stagnation point flow.

In a potential flow the incompressible Bernoulli equation is applicable and is used to obtain a relationship between the pressure at any point in the flow and the stagnation point. Thus, we have

$$P_0 - P_i = \frac{1}{2}\rho\big(U_i^2 + V_i^2\big) = \frac{1}{2}K^2(x^2 + y^2) \qquad (2-48)$$

where the zero subscript indicates the stagnation point.

Returning to the viscous flow case, we have that the governing equations for the dynamics of the two-dimensional steady flow of a viscous incompressible constant property fluid are

continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1-49)$$

momentum

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad (1-50a)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \qquad (1-50b)$$

The continuity equation is integrated by introducing a stream function such that

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x} \qquad (2-49)$$

Hence, the momentum equations become

$$\psi_y\psi_{xy} - \psi_x\psi_{yy} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu(\psi_{xxy} + \psi_{yyy}) \qquad (2-50)$$

and

$$\psi_y\psi_{xx} - \psi_x\psi_{xy} = \frac{1}{\rho}\frac{\partial P}{\partial y} + \nu(\psi_{xxx} + \psi_{xyy}) \qquad (2-51)$$

The appropriate boundary conditions at the plate surface, i.e., at $y = 0$, are $u = v = 0$ or $\psi_y = \psi_x = 0$. At the stagnation point, i.e., at $x = 0$, $y = 0$, we have $P = P_0$. Recall that at a large distance above the plate it was assumed that the inviscid flow is recovered. Hence, as $y \to \infty$, $u \to U_i$, $v \to V_i$. Thus, $u \to Kx$ and $v \to -Ky$, and the pressure field approaches the inviscid pressure field given by Eq. (2–48).

Notwithstanding our experience with the suddenly accelerated flat plate, we seek a solution by separation of variables. Thus, we assume

$$\psi(x, y) = F(x)G(y) = KxG(y) \qquad (2-52)$$

where K is a constant. Assuming a solution of this form is essentially equivalent to predetermining the solution as a function of x. This form of the solution is justified on the basis of the required behavior at infinity, i.e., as $y \to \infty$, $u = \psi_y = KxG'(y) \to Kx$. Here the prime denotes differentiation with respect to the appropriate argument. Our result also implies that as $y \to \infty$, $G'(y) \to 1$. Since Eqs. (2–50) and (2–51) represent two equations in the two dependent variables ψ and P, we must also make an assumption about the form of the viscous pressure field. Assuming that the functional form of the viscous pressure field is closely approximated by that for the inviscid pressure field, we take†

$$P_0 - P = \frac{1}{2}\rho K^2\big(x^2 + F(y)\big) \qquad (2-53)$$

Substitution of Eqs. (2–52) and (2–53) into the governing differential equations Eqs. (2–50) and (2–51) yields

$$K^2xG'G' - K^2xGG'' = K^2x + \nu KxG'''$$

and

$$-K^2GG' = -\frac{K^2}{2}F' + \nu KG''$$

where here the prime denotes differentiation with respect to the argument y. Dividing the first of these equations through by K^2x and the second by $-K^2$ yields

$$G'G' - GG'' = 1 + \frac{\nu}{K}G''' \qquad (2-54)$$

and

$$GG' = \frac{1}{2}F' - \frac{\nu}{K}G'' \qquad (2-55)$$

Immediately we note that the governing partial differential equations have been reduced to ordinary differential equations. Since no equation which is a function of x results, we know that we have correctly chosen the functional form with respect to x in Eqs. (2–52) and (2–53).

From Eq. (2–52) the boundary conditions at $y = 0$ are

$$G(0) = G'(0) = 0 \qquad (2-56)$$

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†The rationale for this assumption is illustrated by assuming

$$P_0 - P = \frac{1}{2}\rho(u^2 + v^2)$$

Substituting for u and v as obtained from Eq. (2–52), we have

$$P_0 - P = \frac{1}{2}\rho K^2\left[(xG')^2 + G^2\right]$$

Realizing that as $y \to \infty$ $G' \to 1$, we finally assume for simplicity that the pressure field has the form given by Eq. (2–53). However, note that $F(y)$ is taken as an arbitrary function of y for greater generality.

At $x = 0$, $y = 0$, i.e., at the stagnation point, from Eq. (2–53) we have

$$P_0 - P = \frac{1}{2}\rho K\big(F(0)\big) = 0$$

Since at $x = 0, y = 0,\ P = P_0$

$$F(0) = 0 \tag{2-57}$$

Recalling that at a large distance above the plate it is assumed that the inviscid flow solution is recovered, we write

$$u = \psi_y = KxG' \to U \to Kx$$

Hence as $y \to \infty$

$$G'(y) \to 1 \tag{2-58}$$

Although an analytical solution of the two-point asymptotic boundary value problem given by Eqs. (2–54) to (2–58) does not presently exist, exact numerical solutions can be obtained. However, to obtain any insight into the results, it is necessary to parameterize the solutions with respect to ν/K. This is inconvenient and also expensive in terms of computation time. Hence, before proceeding further we look for affine stretching transformations for the dependent and independent variables which remove the factor ν/K, i.e., we nondimensionalize the equations. Thus, we let

$$G(y) = \alpha f(\eta) \tag{2-59a}$$

$$F(y) = \gamma g(\eta) \tag{2-59b}$$

$$\eta = \beta y \tag{2-59c}$$

where α, γ, and β are nonzero constants.

Substituting into Eqs. (2–54) and (2–55) yields

$$f'^2 - ff'' = \frac{1}{(\alpha\beta)^2} + \frac{\nu}{K}\frac{\beta}{\alpha}f''' \tag{2-60}$$

$$ff' = \frac{1}{2}\frac{\gamma}{\alpha^2}g' - \frac{\nu}{K}\frac{\beta}{\alpha}f'' \tag{2-61}$$

Before choosing α, γ and β we look at the boundary conditions. The reason is that a particular choice of α, γ and β which eliminates ν/K from the differential equation may result in boundary conditions involving ν/K. Under these circumstances, the solution requires parameterization with respect to the boundary conditions. Transformation of Eqs. (2–56) to (2–58) at $\eta = 0$ yields

$$f(0) = f'(0) = g(0) = 0 \tag{2-62}$$

and as $\eta \to \infty$

$$f'(\eta) = \frac{1}{\alpha\beta} \qquad (2-63)$$

From an examination of Eqs. (2–60), (2–61) and (2–63) we see that an appropriate choice is $\alpha\beta = 1$, $\beta/\alpha = K/\nu$ and $\gamma = 2\alpha^2$. After solving for α, β, γ we have

$$\beta = \left(\frac{K}{\nu}\right)^{1/2} \qquad \alpha = \left(\frac{\nu}{K}\right)^{1/2} \qquad \gamma = 2\frac{\nu}{K} \qquad (2-64)$$

and the governing two-point asymptotic boundary value problem is

$$f''' + f'' + (1 - f'^2) = 0 \qquad (2-65)$$

$$g' = f'' + ff' \qquad (2-66)$$

with boundary conditions

$$\eta = 0 \quad f(0) = f'(0) = g(0) = 0 \qquad (2-67)$$

$$\eta \to \infty \quad f'(\eta) \to 1 \qquad (2-68)$$

Here, we see that not only are the momentum equations uncoupled from the energy equation but in addition the x and y momentum equations, Eqs. (2–65) and (2–66), respectively, are also uncoupled. Thus, we can solve the x momentum equation, Eq. (2–65), independently of the y momentum equation, Eq. (2–66), and subsequently use the solution of Eq. (2–65) in obtaining that of Eq. (2–66). That is, we use the values of $f'(\eta)$ and $f''(\eta)$ obtained from a solution of Eq. (2–65) in Eq. (2–66) to solve for $g(\eta)$.

Equation (2–66) can be directly integrated to yield

$$g = \frac{f^2}{2} + f' + \text{constant}$$

From the boundary conditions given in Eq. (2–56) we see that the constant of integration is zero, hence

$$g = \frac{f^2}{2} + f' \qquad (2-69)$$

No known closed-form analytical solution of the remaining two-point asymptotic boundary value problem is available. Thus, we look for an exact numerical solution. The procedure is to seek an exact solution of Eq. (2-65) and to subsequently use that solution to obtain $g(\eta)$.

Equation (2–65) is a third-order nonlinear[†] ordinary differential equation. Numerical integration of this equation requires a knowledge of $f(0)$, $f'(0)$ and $f''(0)$

[†] A di erential equation is nonlinear if powers and/or products of the dependent variable and/or its derivatives occur; e.g., $y(dy/dx) + y = 0$ is nonlinear, while $x(dy/dx) + y = 0$ is linear.

to start the integration. However, from the given boundary conditions only $f(0)$ and $f'(0)$ are known. The third required boundary condition is specified at infinity. The procedure is to estimate the unknown value of $f''(0)$ and then perform the numerical integration out to some large value of η which we call $\eta_{\max}$. $\eta_{\max}$ is taken to be equivalent to infinity. When the integration has proceeded to $\eta_{\max}$ the value of $f'(\eta_{\max})$ is compared with the required asymptotic value of one. If $f'(\eta_{\max})$ is within some specified small value of one then the outer boundary condition is said to be satisfied and we have a solution of the governing two-point asymptotic boundary value problem. If not, we estimate a new value of $f''(0)$ and repeat the procedure. Since $\eta_{\max} \neq \infty$, $f'(\eta_{\max})$ cannot equal one precisely. Thus, we consider the outer boundary condition to be satisfied if

$$f'(\eta_{\max}) = 1 \pm \epsilon_1 \qquad (2-70)$$

where ϵ_1 is some small quantity, say 5×10^{-7}. This is the so-called 'shooting' method.

2–8 Iteration Scheme

In general the first estimate for $f''(0)$ does not yield a solution. Arbitrary guessing of subsequent estimates of $f''(0)$, of course, proves to be quite inefficient. Hence, a logical method of determining the new estimates for $f''(0)$ must be used. The Newton-Raphson method (see Appendix B) is frequently used for estimating the unknown gradients needed to obtain numerical solutions to linear and nonlinear differential equations. Although the Newton-Raphson iteration scheme assures convergence to the required outer boundary condition at $\eta_{\max}$, it does not insure asymptotic convergence to the specified outer boundary condition required for this two-point asymptotic boundary value problem. Before continuing the discussion, recall that asymptotic convergence implies that as $f'(\eta) \to 1$ its first derivative approaches zero, i.e., $f''(\eta) \to 0$ as $\eta \to \eta_{\max}$. Considering the Newton-Raphson iteration scheme in this context reveals that it does not insure asymptotic convergence; in fact, the Newton-Raphson iteration scheme might be satisfied by $f'(\eta_{\max}) = 1 + \epsilon$ and (say) $f''(\eta_{\max}) = 0.5$. Hence, at $\eta_{\max} + \Delta\eta$, $f'(\eta)$ would not satisfy the outer boundary condition. In addition, if the initial estimate of $f''(0)$ is very far from the correct value the solutions tend to diverge. Under these circumstances the derivatives required in the Newton-Raphson method cannot be calculated in any meaningful manner.

In order to eliminate the problems associated with the Newton-Raphson iteration scheme, the Nachtsheim-Swigert [Nach65] iteration scheme is used. This technique is fully discussed in Appendix B. In summary the Nachtsheim-Swigert iteration scheme is so structured that asymptotic convergence to the correct outer boundary conditions is assured. This is accomplished by imposing the additional condition that

$$f''(\eta_{\max}) = \epsilon_2 \qquad (2-71)$$

where ϵ_2 is some small quantity. The new estimates of $f''(0)$ are obtained such that the sum of the squares of the errors, i.e., ${\epsilon_1}^2 + {\epsilon_2}^2$, in satisfying the asymptotic boundary condition is a minimum. Thus, convergence is achieved in a least squares sense.

A numerical algorithm to solve boundary value problems is discussed in detail in Appendix D.

2–9 Numerical Solution

The results of a typical run of the `stag2d` program described in Section D–12 are shown below.

```
f″(0) = 1.25
η      f″           f′        f
0      1.25         0         0
6.     0.222783     1.57637   6.42524
f″(0) = 1.251
0      1.251        0         0
6.     0.237166     1.61137   6.48878
f″(0) = 1.23368
0      1.23368      0         0
6.     1.24093e−2   1.03421   5.41737
f″(0) = 1.23272
0      1.23272      0         0
6.     1.45136e−3   1.00402   5.35978
f″(0) = 1.2326
0      1.2326       0         0
6.     1.75541e−4   1.00049   5.35303
f″(0) = 1.23259
0      1.23259      0         0
6.     2.1323e−5    1.00006   5.35221
f″(0) = 1.23259
0      1.23259      0         0
6.     2.60013e−6   1.00001   5.35211
f″(0) = 1.23259
0      1.23259      0         0
6.     3.25885e−7   1.        5.3521
Convergence achieved
```

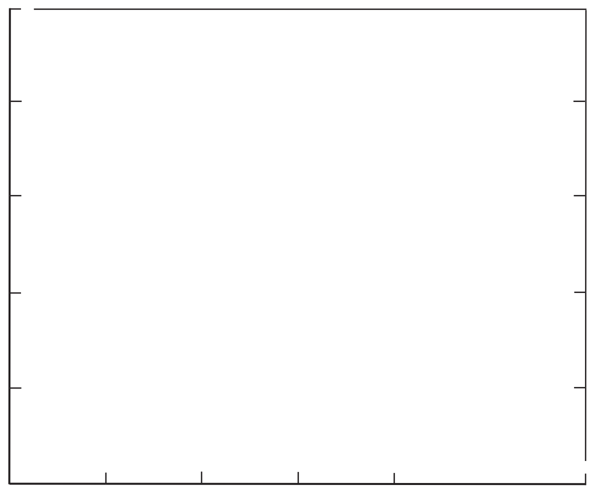
Table 2–1. Solutions for two-dimensional stagnation point flow.

η	f″	f′	f	g
0	1.232588	0	0	0
0.2	1.03445	0.226612	2.33223e−2	0.226884
0.4	0.846325	0.414456	8.80566e−2	0.418333
0.6	0.675171	0.566281	0.186701	0.583709
0.8	0.525131	0.685937	0.312423	0.734742
1.	0.398013	0.777865	0.459227	0.88331
1.2	0.293776	0.846671	0.622028	1.04013
1.4	0.211003	0.896809	0.796652	1.21414
1.6	0.147351	0.932348	0.979779	1.41233
1.8	9.99638e−2	0.956834	1.16886	1.63995
2.	6.58254e−2	0.973217	1.36197	1.9007
2.2	4.20396e−2	0.983853	1.55776	2.19716
2.4	2.60203e−2	0.990549	1.75525	2.53101
2.6	1.55973e−2	0.994634	1.95381	2.90331
2.8	9.04887e−3	0.997046	2.153	3.31474
3.	5.07797e−3	0.998424	2.35256	3.76569
3.2	2.7549e−3	0.999186	2.55233	4.25637
3.4	1.44421e−3	0.999593	2.75221	4.78692
3.6	7.31269e−4	0.999803	2.95215	5.3574
3.8	3.57497e−4	0.999908	3.15212	5.96784
4.	1.6868e−4	0.999958	3.35211	6.61828
4.2	7.67926e−5	0.999982	3.5521	7.3087
4.4	3.37239e−5	0.999992	3.7521	8.03912
4.6	1.42847e−5	0.999997	3.9521	8.80954
4.8	5.83719e−6	0.999999	4.1521	9.61997
5.	2.30344e−6	1.	4.3521	10.4704
5.2	8.80753e−7	1.	4.5521	11.3608
5.4	3.29701e−7	1.	4.7521	12.2912
5.6	1.24507e−7	1.	4.9521	13.2616
5.8	5.12017e−8	1.	5.1521	14.2721
6.	2.623e−8	1.	5.3521	15.3225

Here the initial guess for the unknown initial condition $f''(0)$ for the x momentum equation (Eq. 2–65) is taken as 1.25.† After perturbing the initial guess for $f''(0)$ in order to calculate the Nachtsheim-Swigert iteration derivatives at the edge of the boundary layer, convergence within the required error values of $f''(\eta_{\max}) = \pm 1 \times 10^{-6}$ and $f'(\eta_{\max}) - 1 = \pm 1 \times 10^{-6}$ occurs after six iterations.

Solution of the y momentum equation is obtained by using these results in Eq. (2–69). The results are tabulated in Table 2–1 and shown in Figure 2–7.

†Since no prior knowledge of the solution of equations of this type was available, the value of $f''(0)$ = 1.25 was arrived at by initially limiting the value of $\eta_{\max}$ as discussed in Appendix B.



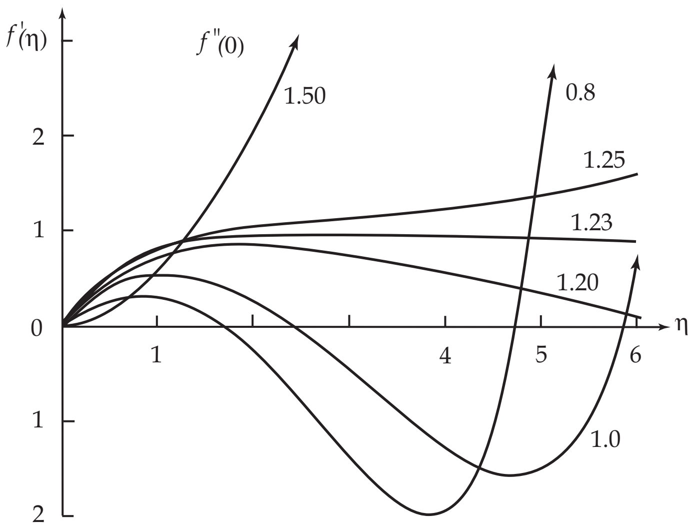


Figure 2–8. E ect of $f''(0)$ on $f'(\eta)$.

is somewhat improved in this case. These modifications are left as an exercise for the reader (see Problem 2–5).

2–10 **Axisymmetric Stagnation Point Flow**

The previously obtained solution for two-dimensional stagnation point flow can be extended to the case of an axisymmetric stream impinging on a plane wall. The solution obtained is representative of that near the forward stagnation point of an axisymmetric blunt body. Figure 2–9 illustrates the problem under discussion. Here x and y are the radial and axial directions, respectively, with u and v the velocity components in the x and y directions, respectively. The plane is assumed to be perpendicular to the flow direction, with the stagnation point as the center of the coordinate system. The momentum equations for steady axisymmetric incompressible flow with constant properties are obtained by transforming Eqs. (1–49) and (1–50) into cylindrical polar coordinates. They are then (see Goldstein [Gold38], p. 143)

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{x}\frac{\partial u}{\partial x} - \frac{u}{x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad (2-72)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{1}{x}\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial y^2}\right) \qquad (2-73)$$

The axisymmetric continuity equation is

$$\frac{\partial}{\partial x}(xu) + \frac{\partial}{\partial y}(xv) = 0 \qquad (2-74)$$

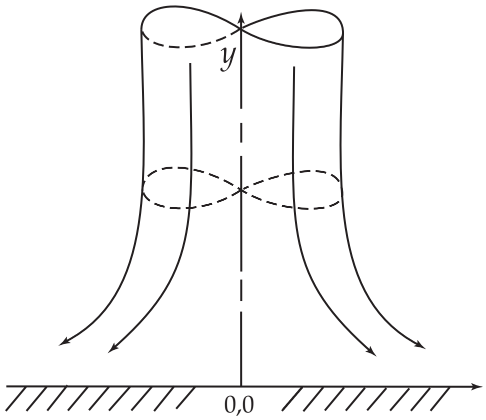


Figure 2–9. Axisymmetric stagnation point flow.

The boundary conditions are again taken to be no-slip at the plate surface and no-mass-transfer through the plate surface. Further, it is required that the inviscid solution be recovered at a large distance from the body. For axisymmetric inviscid stagnation point flow, the velocity components are given by (see Moore [Moor64], p. 79)

$$U_i = Kx \tag{2 − 75}$$

$$V_i = -2Ky \tag{2 − 76}$$

and the pressure distribution by

$$P_0 - P_i = \frac{1}{2}\rho K^2\left(x^2 + 4y^2\right) \tag{2 − 77}$$

where the zero subscript indicates the stagnation point at $x = y = 0$.†

The boundary conditions at the surface become $u = v = 0$ at $y = 0$. Further, at the stagnation point, $y = 0$, $x = 0$, $P = P_0$. At a large distance from the plate, i.e., as $y \to \infty$, $u \to U_i \to Kx$, $v \to V_i \to -2Ky$.

The continuity equation is automatically satisfied by introducing a stream function of the form

$$u = \frac{1}{x}\frac{\partial\psi}{\partial y} \qquad v = -\frac{1}{x}\frac{\partial\psi}{\partial x} \tag{2 − 78}$$

†Alternatively, we may take $U_i = Kx/2$, $V_i = -Kx$, in which case the pressure distribution is

$$P_0 - P_i = \frac{1}{2}\rho K^2\left[\left(\frac{x}{2}\right)^2 + y^2\right]$$

Introducing the stream function into the momentum equations yields

$$\frac{1}{x^2}\psi_y\psi_{xy} - \frac{1}{x^2}\psi_x\psi_{yy} - \frac{1}{x^3}\psi_y^2 = \\ -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{1}{x}\psi_{yyy} + \frac{1}{x}\psi_{xxy} - \frac{1}{x^2}\psi_{xy}\right) \qquad (2-79)$$

and

$$\frac{1}{x^2}\psi_x\psi_{xy} - \frac{1}{x^2}\psi_{xx}\psi_y + \frac{1}{x^3}\psi_x\psi_y = \\ -\frac{1}{\rho}\frac{\partial P}{\partial y} - \nu\left(\frac{1}{x}\psi_{xxx} + \frac{1}{x}\psi_{xyy} + \frac{1}{x^3}\psi_x - \frac{1}{x^2}\psi_{xx}\right) \qquad (2-80)$$

By analogy with two-dimensional stagnation point flow, we seek solutions of the viscous momentum equations by separation of variables. To this end we assume that

$$\psi(x,y) = Kx^aG(y) \qquad (2-81)$$

where a is an as yet undetermined constant. Choosing this form for $\psi(x,y)$ presupposes that the x dependence is a power law. We again assume that

$$P_0 - P = \frac{1}{2}\rho K^2\big[x^2 + F(y)\big] \qquad (2-82)$$

Substituting into the momentum equations, the x momentum equation is

$$(a-1)K^2x^{2a-3}G'^2 - aK^2x^{2a-3}GG'' = \\ K^2x + \nu\big[Kx^{a-1}G''' + a(a-2)Kx^{a-3}G'\big] \qquad (2-83)$$

and the y momentum equation becomes

$$2aK^2x^{2a-4}GG' = \frac{K^2F'}{2} - \nu\big[a(a-2)^2Kx^{a-4}G + aKx^{a-2}G''\big] \qquad (2-84)$$

where here the prime denotes differentiation with respect to the argument y. Inspection of Eqs. (2–83) and (2–84) shows that if $a = 2$ these partial differential equations reduce to ordinary differential equations. Thus, after dividing by K^2 and $4K^2$ in the x and y momentum equations, respectively, we have

$$\frac{\nu}{K}G''' + 2GG'' + \big(1 - G'^2\big) = 0 \qquad (2-85)$$

and

$$GG' = -\frac{F'}{8} - \frac{\nu}{2K}G'' \qquad (2-86)$$

Here, note that Eqs. (2–85) and (2–86) are very similar in form to Eqs. (2–54) and (2–55). Again, the equations are nondimensionalized by seeking affine stretching transformations of the form

$$G(y) = \alpha f(\eta) \tag{2 – 87a}$$

$$F(y) = \gamma g(\eta) \tag{2 – 87b}$$

$$\eta = \beta y \tag{2 – 87c}$$

where α, γ, β are nonzero constants. Substituting into Eqs. (2–85) and (2–86) yields

$$f''' + \frac{2K}{\nu}\frac{\alpha}{\beta} f f'' + \frac{K}{\nu}\frac{1}{\alpha\beta^3}\left[1 - (\alpha\beta)^2 f'^2\right] = 0 \tag{2 – 88}$$

and

$$f f' = \frac{\gamma}{8}\frac{1}{\alpha^2} g' - \frac{\nu}{2K}\frac{\beta}{\alpha} f'' \tag{2 – 89}$$

Before choosing particular values for α, γ and β the boundary conditions must be investigated. The transformed surface boundary conditions are

$$\eta = 0 \qquad f(0) = f'(0) = g(0) = 0 \tag{2 – 90a}$$

The transformed boundary condition at infinity is

$$\eta \to \infty \qquad f(\eta) \to \frac{1}{\alpha\beta} \tag{2 – 90b}$$

Hence, in order to obtain a simplified boundary condition at infinity we take $\alpha\beta = 1$. Finally, we take

$$\frac{2K}{\nu}\frac{\alpha}{\beta} = 1$$

and

$$\gamma = 8\alpha^2$$

Hence

$$\beta = \left(\frac{2K}{\nu}\right)^{1/2} \qquad \alpha = \left(\frac{\nu}{2K}\right)^{1/2} \qquad \gamma = \frac{4\nu}{K} \tag{2 – 91}$$

The governing two-point asymptotic boundary value problem is then given as

$$f''' + f f'' + \frac{1}{2}(1 - f'^2) = 0 \tag{2 – 92}$$

$$g' = f f' + f'' \tag{2 – 93}$$

with boundary conditions

$$\eta = 0 \qquad f(0) = f'(0) = g(0) = 0 \tag{2 – 94}$$

$$\eta \to \infty \qquad f(\eta) \to 1 \tag{2 – 95}$$

As was the case for two-dimensional stagnation point flow, the transformed y momentum equation is immediately integrable. Hence

$$g(\eta) = \frac{f^2}{2} + f' \qquad (2-96)$$

Comparing Eqs. (2–92) and (2–65), we see that they differ only by the factor of one-half multiplying the last term. The solution of Eq. (2–92) is obtained by the modifications to the `eqmot2d` routine used with `stag2d`. Specifically, `eqmot2d` becomes

equations of motion — two-dimensional axisymmetric stagnation point flow

```
subroutine eqmot(eta, x(), param(), f())
f(1) = −x(3)∗x(1) −0.5∗(1 − x(2)∗x(2))
f(2) = x(1)
f(3) = x(2)
return
```

A typical run using the above modified `eqmot2d` routine is

```
f″(0) = 1.23259
η      f″             f′          f
0      1.232588       0           0
6.     0.423879       3.29404     11.6197
f″(0) = 1.233588
0      1.233588       0           0
6.     0.425169       3.30122     11.6401
f″(0) = 0.912905
0        0.912905     0           0
6.     −0.022935      0.881218    4.87142
f″(0) = 0.929481
0      0.929481       0           0
6.     2.7727e−3      1.01441     5.23484
f″(0) = 0.92747
0        0.92747      0           0
6.     −3.22936e−4    0.998322    5.19087
f″(0) = 0.927705
0      0.927705       0           0
6.     3.79112e−5     1.0002      5.19599
f″(0) = 1.23259
```

```
f''(0) = 0.927677

0      0.927677      0          0
6.    -4.31852e-6    0.999977   5.19539

f''(0) = 0.92768

0      0.92768       0          0
6.     6.21323e-7    1.         5.19546

f''(0) = 0.92768

0      0.92768       0          0
6.     4.34516e-8    1.         5.19545

Convergence achieved
```

As shown for two-dimensional stagnation point flow, arbitrary estimation of the value of $f''(0)$ can lead to results for the Nachtsheim-Swigert iteration derivatives, which do not yield convergence to the required outer boundary condition $f'(\eta_{\max}) \to 1$. Hence, it is desirable to have some knowledge of an approximate range of values for $f''(0)$. If either an analytical or exact solution of a boundary value problem similar to that under investigation is known, then it is reasonable to use the value of $f''(0)$ obtained for the known solution as the first estimate for the problem under investigation. Considering the similarity of Eqs. (2–92) and (2–65), the first estimate of $f''(0)$ is taken to be that for two-dimensional stagnation point flow, i.e., 1.232588. Convergence to within $f''(\eta_{\max}) = \pm 1 \times 10^{-6}$ and $f'(\eta_{\max}) - 1 = \pm 1 \times 10^{-6}$ in seven iterations yields a value of $f''(0) = 0.927680$. Note that the correction value for the last iteration is less than 5×10^{-6}. The complete solution including the y momentum equation (see Eq. 2–96) is shown in Table 2–2. The value of $f''(0) = 0.927680$ agrees with that originally calculated by Homann [Homa36].

The nondimensional velocity and pressure function results for axisymmetric stagnation point flow are shown graphically in Figure 2–10. Also shown are the results for two-dimensional stagnation point flow.

From an analysis of the previous assumptions and transformations we see that for both two-dimensional and axisymmetric flow the local velocity components and the stream function are related to their respective inviscid values by

$$\frac{u}{U_i} = f'(\eta) \qquad \frac{v}{V_i} = -\frac{f'(\eta)}{\eta} \qquad \frac{\psi}{\psi_i} = \frac{f(\eta)}{\eta} \qquad (2-97a,b,c)$$

Further, the pressure field for two-dimensional viscous stagnation point flow is

$$P_0 - P = \frac{\rho}{2}K^2\left(x^2 + \frac{2g(\eta)}{\eta^2}y^2\right) \qquad (2-98)$$

and that for axisymmetric viscous stagnation point flow

$$P_0 - P = \frac{\rho K^2}{2}\left(x^2 + 4\frac{2g(\eta)}{\eta^2}y^2\right) \qquad (2-99)$$

Table 2–2. Solutions for axisymmetric stagnation point flow.

η	f″	f′	f	g
0	0.92768	0	0	0
0.2	0.82771	0.175537	0.017887	0.175697
0.4	0.728152	0.33111	6.88836e−2	0.333483
0.6	0.630002	0.466891	0.149011	0.477993
0.8	0.53477	0.583305	0.254348	0.615651
1.	0.444284	0.681115	0.381092	0.753731
1.2	0.360449	0.761462	0.525629	0.899605
1.4	0.284977	0.825853	0.684612	1.0602
1.6	0.219151	0.876098	0.855027	1.24163
1.8	0.163652	0.914204	1.03424	1.44903
2.	0.1185	0.94225	1.22004	1.6865
2.2	8.30998e−2	0.962255	1.41061	1.95716
2.4	0.056379	0.976069	1.60453	2.26332
2.6	3.69739e−2	0.985294	1.80073	2.60661
2.8	2.34217e−2	0.991248	1.99843	2.9881
3.	1.43227e−2	0.994959	2.19708	3.40854
3.2	8.4508e−3	0.997192	2.39631	3.86835
3.4	4.80892e−3	0.998488	2.59589	4.36782
3.6	2.63824e−3	0.999213	2.79567	4.9071
3.8	1.39496e−3	0.999605	2.99556	5.48628
4.	7.10674e−4	0.999808	3.1955	6.10542
4.2	3.48767e−4	0.99991	3.39547	6.76453
4.4	1.6484e−4	0.999959	3.59546	7.46363
4.6	7.50188e−5	0.999982	3.79546	8.20272
4.8	3.2869e−5	0.999993	3.99545	8.98181
5.	1.38626e−5	0.999997	4.19545	9.80091
5.2	5.62723e−6	0.999999	4.39545	10.66
5.4	2.19832e−6	1.	4.59545	11.5591
5.6	8.26459e−7	1.	4.79545	12.4982
5.8	2.99049e−7	1.	4.99545	13.4773
6.	1.04212e−7	1.	5.19545	14.4964

Referring to Eqs. (2–48) and (2–77) shows that the viscous pressure fields are given by similar modifications of the inviscid pressure fields. Looking at the tabulated results, we see that close to the plate surface $2g(\eta)/\eta^2 > 1$ and hence the local pressure P is larger than in the inviscid case, whereas far from the plate, i.e., for larger values of η, $2g(\eta)/\eta^2 < 1$. Hence, the local pressure P is less than in the inviscid case. Further, the effect is less for axisymmetric than for two-dimensional viscous stagnation point flow.

Returning to the stream function and velocity components, we see from the numerical results that the viscous stream function and the velocity components

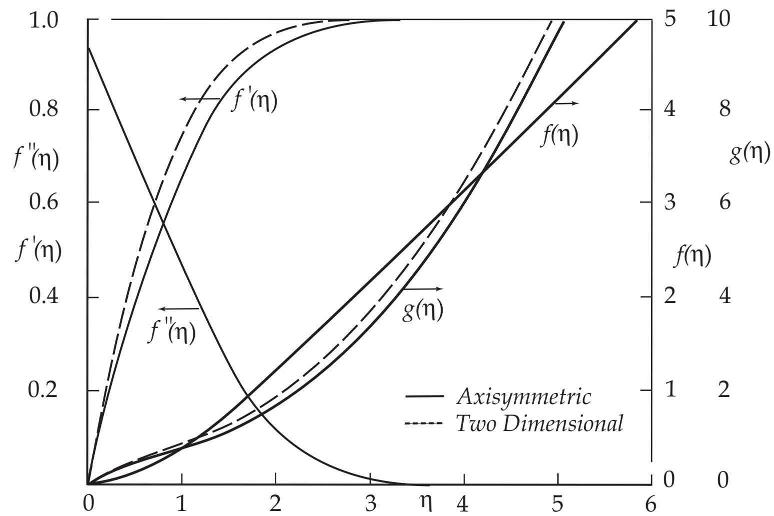


Figure 2–10. Nondimensional results for axisymmetric stagnation point flow.

are always less than the corresponding inviscid results. Further, the effect is greater for axisymmetric than for two-dimensional stagnation point flow.

Finally, we investigate the shearing stress at the wall. From the results given in Chapter 1, we see that the shearing stress at the wall for both axisymmetric and two-dimensional stagnation point flow is given by

$$\tau = \mu \frac{\partial u}{\partial y}\Big|_{y=0} \tag{2 – 100}$$

In transformed coordinates, the shearing stress at the wall for both two-dimensional and axisymmetric stagnation point flow is

$$\tau = \mu K x f''(0) \tag{2 – 101}$$

Hence, the shearing stress at the wall is greater for two-dimensional stagnation point flow than for axisymmetric stagnation point flow.

The technique developed in this chapter for solving an asymptotic two-point boundary value problem is useful in succeeding chapters, where solutions for the velocity profiles in laminar boundary layer flows are obtained.