APPENDIX E

PROBLEMS

The problems given here are generally of three types: numerical, parametric and directed analysis. The numerical problems are generally of the form — 'Given the following specific information, calculate a required result'. This is the type of problem generally found in textbooks. The parametric problems are of the form — 'Investigate the effect of the variation of a parameter on the numerical solution'. These problems generally are amenable to solutions using programs derived from the pseudocode algorithms and/or modification of these algorithms. They allow the student to ask the question 'What if —— ?' and lead to a better physical understanding of a mathematical model. Such problems are useful in teaching the concept of parametric analysis. Directed analysis problems seek to guide the student through an original analysis of a given problem. They are frequently skeletons of the analyses in classical papers. They serve to extend and diversify the material and to tailor the material to individual interests and needs. For example, there are several directed analysis problems on rotational flows which can serve as an introduction to that topic. Frequently, the result of a directed analysis problem is a differential equation or system of differential equations that require numerical solution. Here, the reader is shown how to put these equations in a form similar to that of the equations solved by one of the analysis algorithms. Consequently, the directed analysis problems serve as a unifying mechanism for various topics in fluid dynamics. Directed analysis problems generally require considerable effort. Problems that require or lead to computer solutions are marked with (c).

CHAPTER 2

2–1(c) Numerically integrate the governing boundary value problem for the suddenly accelerated flat plate. (Hint: Modify the `stag2d` algorithm, and reduce the order of the system.) Compare the results with those for the analytical solution.

2–2 Using Eqs. (2–52) and (2–53), derive Eqs. (2–54) to (2-58).

2–3 Using Eq. (2–78), derive Eqs. (2–79) and (2-80).

2–4 Using Eqs. (2–81) and (2–82), derive Eqs. (2-83) and (2–84).

2–5(c) Modify the `stag2d` program to automatically increment $\eta_{\max}$ as the iteration proceeds (see Appendix B and Figure 2–8).

2–6(c) Modify the `ns` routine of Appendix D to dynamically update the Nachtsheim-Swigert iteration derivative at the edge of the boundary layer for the `stag2d` program.

2–7 Consider the starting process in simple Couette flow. The flow is considered to be two-dimensional, incompressible and without body forces. The governing equations are then

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

with boundary conditions

$$u = 0 \quad \text{for} \quad 0 \le y \le h \qquad t \le 0$$

$$\left.\begin{array}{ll} u = 0 & y = 0 \\ u = U & y = h \end{array}\right\} \quad t > 0$$

a. Using a transformation of the form

$$u = Af(\eta) \qquad \eta = Bt^n y^m$$

show that the governing equation reduces to the linear ordinary differential equation

$$C_1 f'' + C_2 \eta f' = 0$$

with boundary conditions of the form

$$f(0) = C_3 \qquad f(C_4 t) = C_5 g(t)$$

where the C_n are constants and the prime denotes differentiation with respect to η.

b. Since the resulting governing equation is linear, use the method of superposition to obtain a solution. Hint: the resulting solution is a series solution of the form

$$f = \sum_{n=0}^{\infty} F_n(\eta + C_n \eta_1)$$

c. Plot the velocity profile $u(y,t)$ vs y for several values of time $t > 0$. Show that as $t \to \infty$ the profile approaches the linear profile associated with simple steady Couette flow.

2–8 Consider the flow between two infinite concentric rotating cylinders of radius r_1 and r_2, with angular velocities ω_1 and ω_2, respectively. Assume that the flow is steady, two-dimensional and the fluid viscous and incompressible.

a. Show that the incompressible Navier-Stokes equations in cylindrical coordinates, i.e.

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} + u_z\frac{\partial u_r}{\partial z}\right) = F_r - \frac{\partial P}{\partial r}$$
$$+\mu\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial^2 u_r}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} - \frac{u_r u_\theta}{r} + u_z\frac{\partial u_\theta}{\partial z}\right) = F_\theta - \frac{1}{r}\frac{\partial P}{\partial \theta}$$
$$+\mu\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} + \frac{\partial^2 u_\theta}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z}\right) = F_z - \frac{\partial P}{\partial z}$$
$$+\mu\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2}\right)$$

with the continuity equation given as

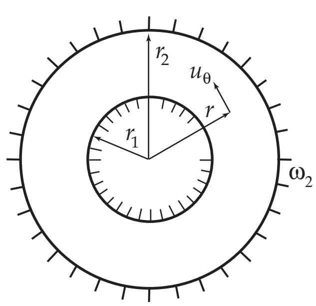
$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

where r, θ and z are the radial, azimuthal and axial coordinates, respectively; u_r, u_θ, u_z are the velocity components in the r, θ, z directions, respectively; and F_r, F_θ, F_z are the r, θ, z components of the body force/unit mass, respectively, reduce to

$$\rho\frac{u_\theta^2}{r} = \frac{dP}{dr} \tag{1}$$

and

$$\frac{d^2 u_\theta}{dr^2} + \frac{d}{ddr}\left(\frac{u_\theta}{r}\right) = 0 \tag{2}$$



b. Assuming that the no-slip condition is valid at the surfaces of both cylinders, show that the velocity distribution is

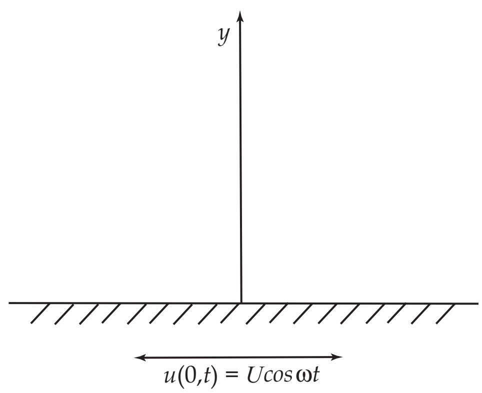
$$u_\theta(r) = \frac{1}{r_2^2 - r_1^2}\left[r(\omega_2 r_2^2 - \omega_1 r_1^2) - \frac{r_1^2 r_2^2}{r}(\omega_2 - \omega_1)\right]$$

c. Use the radial velocity distribution to determine the radial pressure distribution.

d. Using Newton's law of friction, determine the shearing stress at the surface of each cylinder.

e. Assuming that the inner cylinder is at rest, i.e., $\omega_1 = 0$, determine the torque transmitted to the fluid by the rotation of the outer cylinder. Assume that the cylinders are of height h.

2–9 Consider the flow about an infinite flat plate undergoing harmonic oscillations in the plane of the plate. The fluid is viscous and incompressible.



a. Show that the governing equation is

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

b. Assume that the no-slip boundary condition applies at the plate and that the motion of the plate is given by $u(0,t) = U \cos \omega t$. Since the motion of the plate is harmonic, it seems reasonable to expect the velocity of the fluid to vary harmonically with time. Since the fluid has viscosity, it is reasonable to expect the fluid motion to exhibit attenuation and phase shift with respect to the motion of the plate which logically depend on the distance above the plate. Thus, assume a solution of the form

$$u(y,t) = Ae^{a\eta} \cos(wt - \eta)$$

$$\eta = by$$

and determine the constants A, a, b such that the boundary conditions are satisfied.

c. Introduce appropriate nondimensional dependent and independent variables, f and η, and plot the velocity profile, $f(\eta, t)$, for several values of ωt, say 0, $\pi/2$, π, $3\pi/2$, 2π. Discuss the phase relationship between the fluid motion and the motion of the plate.

2–10 Consider the flow emerging from a point source into a quiescent fluid. Assume that the flow is viscous, incompressible, steady and axially symmetric. Using spherical polar coordinates (r, θ, ϕ), with θ measured from the axis of the jet, and u_r, u_θ, u_ϕ, the velocity components in the r, θ, ϕ directions, respectively, show that an exact solution of the Navier-Stokes equations can be obtained. Since the jet is axisymmetric, $u_\phi = 0$, and all quantities are independent of ϕ. Thus, the Navier-Stokes equations, i.e.

$$\rho\left[\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{u_\theta^2 + u_\phi^2}{r}\right] = F_r - \frac{\partial P}{\partial r}$$
$$+\mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_r}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_r}{\partial\theta}\right) + \frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2 u_r}{\partial\phi^2}\right.$$
$$\left. - \frac{2u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta} - \frac{2u_\theta\cot\theta}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right]$$

$$\rho\left[\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_\theta}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \frac{u_r u_\theta}{r} - \frac{u_\phi^2\cot\theta}{r}\right] = F_\theta - \frac{1}{r}\frac{\partial P}{\partial \theta}$$
$$+\mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\theta}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_\theta}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\theta}{\partial\phi^2}\right.$$
$$\left. + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\phi}{\partial\phi}\right]$$

$$\rho\left[\frac{\partial u_\phi}{\partial t} + u_r\frac{\partial u_\phi}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\phi}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_\phi u_r}{r} + \frac{u_\theta u_\phi\cot\theta}{r}\right] = F_\phi$$
$$- \frac{1}{r\sin\theta}\frac{\partial P}{\partial\phi} + \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_\phi}{\partial\theta}\right)\right.$$
$$\left. + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\phi}{\partial\phi^2} - \frac{u_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin^2\theta}\frac{\partial u_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\phi}\right]$$

and the continuity equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(u_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} = 0$$

are considerably simplified.

a. Obtain the simplified governing equations.
b. Since there is no characteristic length in this flow, assume a solution of the form

$$\psi(r,\theta) = Ar^a f(\eta)$$
$$\eta = B(\cos\theta)^b$$
$$\frac{P - P_\infty}{\rho} = Cr^c P(\eta)$$

where P_∞ is the pressure at infinity where the velocity components vanish.

1. Show that the continuity equation is satisfied if we assume

$$u_r = +Ar^{-a}f'(\eta) \qquad u_\theta = -\frac{Ar^{-a}f(\eta)}{\sqrt{1-\eta^2}}$$

What are B and b?

2. Show that the θ momentum equation can be reduced to

$$P' = -\left[\frac{f^2}{2(1-\eta^2)}\right]' - f''$$

where the prime denotes differentiation with respect to η.

3. Show that the r momentum equation reduces to

$$P' = \frac{-f^2}{2(1-\eta^2)} - \frac{1}{2}[ff' - (1-\eta^2)f'']'$$

4. What are A, B, C and a, b, c?

c. First integrate the θ momentum equation, and then use this result to integrate the r momentum equation to give

$$f^2 = 2(1-\eta^2)f' + 4\eta f + C_1\eta^2 + C_2\eta + C_3$$

where C_1, C_2, C_3 are constants of integration.

d. Restricting the discussion to the case $C_1 = C_2 = C_3 = 0$, show that

$$f = \frac{2(1-\eta^2)}{\bar{a} + (1-\eta)}$$

where $\bar{a}$ is a constant of integration.

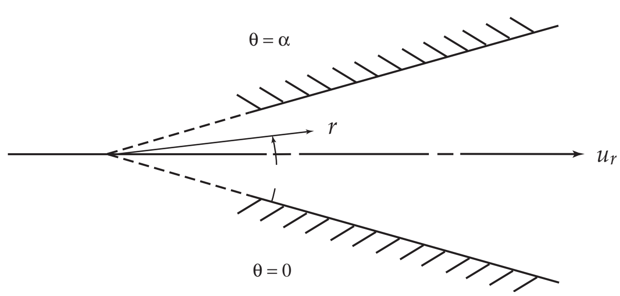
e. Plot the streamlines for $\bar{a} = 10$, 0.1 and 0.01, i.e., plot

$$\psi = Ar^a f(\eta) \quad \text{for} \quad \bar{a} = 10, 0.1 \text{ and } 0.01$$

and show that the stream tubes have a minimum throat area for

$$\eta = \frac{1}{1+\bar{a}}$$

2–11 Consider two-dimensional steady incompressible viscous flow without body forces in a channel with nonparallel straight walls, i.e., flow in a convergent or divergent channel.



a. Using a substitution of the form

$$u(r,\theta) = Ar^a f(\eta)$$

$$\eta = Br^b\theta^t$$

determine the functional form of $u(r,\theta)$ which satisfies the continuity equation.

b. Show that the Navier-Stokes equations in cylindrical polar coordinates (see Problem 2–6) reduce to

$$u_r\frac{\partial u_r}{\partial r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \nu\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} - \frac{u_r}{r^2}\right)$$

and

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial \theta} + \frac{2\nu}{r}\frac{\partial u_r}{\partial \theta}$$

c. Eliminate the pressure from the Navier-Stokes equations, and use the results of (a) above to show that the problem is reduced to the solution of an ordinary differential equation of the form

$$f''' + C_1 f f' + C_2 f' = 0$$

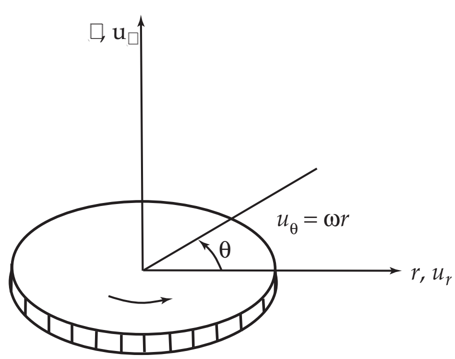
where C_1 and C_2 are constants.

d. Show that the boundary conditions given by the no-slip condition at the surfaces, i.e., $u_r = 0$ at $\theta = 0$ and at α, plus the auxiliary condition that the velocity profiles be symmetrical about the center line, i.e., $\partial u_r/\partial r = 0$ at $\theta = \alpha/2$, is reduced to

$$f(0) = 0 \qquad f(2) = 0 \qquad f'(1) = 0$$

e. (c) This ordinary differential equation, subject to these boundary conditions, can be integrated analytically in terms of elliptic functions (see [Rose63], p. 144). However, the solution is somewhat involved. Therefore, consider a numerical solution. Noting that the velocity profile is symmetrical about $\eta = 1$, only the region $0 \le \eta \le 1$ need be considered. The integration problem is thus similar to that encountered for stagnation point flow. Modify the `stag2d` algorithm, and obtain solutions to the present governing equations. Plot and discuss these solutions.

2–12 Consider the viscous incompressible steady flow without body forces about a flat disk rotating in its own plane with a constant angular velocity, ω. The surrounding fluid is at rest.



a. Show that the incompressible Navier-Stokes equations in cylindrical polar coordinates (see Problem 2–6) reduce to

$$u_r \frac{\partial u_r}{\partial r} - \frac{u_\theta^2}{r} + u_z \frac{\partial u_r}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \nu\left\{\frac{\partial^2 u_r}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{u_r}{r}\right) + \frac{\partial^2 u_r}{\partial z^2}\right\}$$

$$u_r \frac{\partial u_\theta}{\partial r} + \frac{u_r u_\theta}{r} + u_z \frac{\partial u_\theta}{\partial z} = \nu\left\{\frac{\partial^2 u_\theta}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{\partial^2 u_\theta}{\partial z^2}\right\}$$

$$u_r \frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\left\{\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{\partial^2 u_z}{\partial z^2}\right\}$$

with the continuity equation given by

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} = 0$$

with the no-slip boundary conditions

$$z = 0 \qquad u_r = 0 \qquad u_\theta = \omega r \qquad u_z = 0$$

$$z \to \infty \qquad u_r \to 0 \qquad u_\theta \to 0$$

Note that this is a truly three-dimensional flow.

b. Using transformations of the form

$$u_r = A r^a \omega^m F(\eta)$$

$$u_\theta = B r^b \omega^n G(\eta)$$

$$u_z = C r^c \omega^q H(\eta)$$

$$P = D\rho r^d \omega^s P(\eta)$$

and

$$\eta = E r^e \omega^t z^i$$

show that the governing equations can be reduced to the system of simultaneous ordinary differential equations

$$C_1 F + C_2 H' = 0 \qquad (1)$$

$$C_3 F^2 + C_4 F' H + C_5 G^2 + C_6 F'' = 0 \qquad (2)$$

$$C_7 F G + C_8 H G' + C_9 G'' = 0 \qquad (3)$$

$$C_{10} P' + C_{11} H H' + C_{12} H'' = 0 \qquad (4)$$

with boundary conditions

$$\eta = 0 \qquad F = 0 \qquad G = 1 \qquad H = 0 \qquad P = 0$$

$$\eta \to \infty \qquad F \to 0 \qquad G \to 0$$

where the C_n are integer constants *generally* $= \pm 1$ or ± 2, and the prime denotes differentiation with respect to η. Note: be careful in choosing a, $b \ldots$ etc., not to eliminate one of the independent variables.

c. Using Eq. (a), show that the solution can be reduced to the integration of a pair of coupled ordinary differential equations of the form

$$C_{13}H''' + C_{14}HH'' + C_{15}G^2 + C_{16}H'^2 = 0$$

and

$$C_{17}G'' + C_{18}(HG)' = 0$$

with boundary conditions

$$\eta = 0 \qquad H = 0 \qquad H' = 0 \qquad G = 1$$
$$\eta \to \infty \qquad H' \to 0 \qquad G \to 0$$

The solutions for $P(\eta)$ and $F(\eta)$ are then obtained sequentially.

This problem is of a form similar to that governing the laminar free convection boundary layer discussed in Chapter 5. Its numerical integration is addressed in Problem 5–1.